Categories and their Algebra

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Abstract

The goal if this paper is to introduce the notion of a category, along with some basic results, and then to explore work done by Bret Tilson in his "Categories as Algebra" [9]. This involves an explanation of Tilson's work extending the division of monoids to a division on categories, and his use of the derived category of a relational morphism, culminating in his Derived Category Theorem. After using these concepts to explore a few aspects of categories and monoids, the final section reviews some of Tilson's work on varieties of monoids and categories, leading to his Strongly Connected Component Theorem.

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1 Introduction

The concept of a category as an algebraic object was introduced by Samuel Eilenberg and Saunders Mac Lane in their 1945 paper "General Theory of Natural Equivalences" [3]. Their motivation was to make rigourous the notion of a *natural transformation*, a relation which connects different kinds of algebraic objects in a "natural" way. This required the authors to define the notion of a functor, which required in turn the notion of a category. Categories concern objects and their morphisms, functors are morphisms of categories, and natural transformations are morphisms of functors.

Eilenberg and Mac Lane see both technical and conceptual advantages to the use of category theory. Regarding the former, category theory "provides the exact hypotheses necessary to apply to both sides on an isomorphism a passage to the limit," and "a technical background for the intuitive notion of naturality" [3, p.236]. Regarding the latter, it "provides general concepts applicable to all branches of abstract mathematics, and so contributes to the current trend towards uniform treatment of different mathematical disciplines," following the tradition of Emmy Noether [3, p.236].

Other authors, such as Lambek and Scott, have proposed that category theory offers even more. They see category theory an alternative to set theory as a foundation for mathematics, and worked out the details of categorical logic [5]. Such work is controversial, and is connected with intuitionistic logic.

Bret Tilson, in his "Categories as Algebra", sees category theory as a powerful tool to extend the theory of monoids [9]. The core of this paper will explore some of Tilson's results. The key concept is that a category with a single object is a monoid, and so categories are a kind of generalized monoid. Tilson extends the notion of monoid division to include categories, and then defines the "derived category" for a relational morphism. His next step is to investigate "varieties" of monoids and categories, and the mathematical structures inherit in them.

The goal of this paper is first to introduce the concept of a category, and then to explain some of Tilson's key results. My approach in the next section is to present categories from three different perspectives. First, categories are a kind of directed graph with additional structure, and graph theory is an intuitive way to approach categories. Second, categories have their own algebraic structure, and can be explained using a few axioms built on set theory. Third, Tilson makes extensive use of the connection between monoids and categories, and so I introduce monoids at the beginning. Then I consider some examples of categories, and close with a discussion of some of the difficulties and possibilities which one finds between set theory and category theory. Lastly, I introduce basic category theory concepts such as the functor and the natural transformation, and some simple results related to them.

The third section is the core of the paper. It follows Tilson as he extends monoid division to include categories, defines a derived category of a relational morphism, and lays out his Derived Category Theorem. This theorem is the main result we will prove, and it leads to some other interesting results about categories, monoids, and graphs.

In the final section we survey some of Tilson's work on varieties of monoids and categories. Varieties are collections which are closed under products and division, so they are another level of abstraction higher. Tilson shows that varieties of categories can be completely specified by their "laws," which can in turn be built up out of "path equations" which govern the graph structure of a category. We end with the Strongly Connected Component Theorem, in which Tilson demonstrates the connection between a category and its strongly connected components.

2 Categories

A category can be defined in several equivalent ways. The first, and perhaps the most intuitive, is as a kind of graph. One can also define categories axiomatically, as Eilenberg and Mac Lane originally did. And categories have many similarities to monoids, so they can also be described in those terms. Because each of these three approaches has its own strengths, the first goal of this paper will be to describe these equivalent notions of a category.

With these three definitions in hand, several examples of categories will be put forward. However, categories are also interesting because they need not be defined directly in terms of sets (although this is the simplest way). After exploring the connection between categories and sets, we discuss the relationships between categories: functors, which connect one category to another, and natural transformations, which relate two functors.

2.1 Categories as Graphs

The first definition of a category which we will explore is in terms of graphs.[1]

[1]Concerning graphs, my primary source for this section has been West's *Introduction to Graph Theory* [10]. For the material on categories as graphs I have used Mac Lane's *Categories for the Working Mathematician*

A *graph* G is composed of a class of *vertices* $\mathrm{V}(G)$, and for each pair of vertices $v, w \in \mathrm{V}(G)$ a set of *edges* staring at v and ending at w. We denote the set of all edges $\mathrm{E}(G)$, and the *edge-set* between a pair of vertices $G(v, w)$. We can also refer to edges with mapping notation, $e : v \to w$.

A *category* C is a graph with two additional conditions. However, we adopt different naming conventions for categories: we call the vertices *objects* and refer to the object set $\mathrm{Obj}(C)$, and we call the edges *arrows* and refer to the set of arrows $\mathrm{Arr}(C)$. We also refer to the edge-sets as *hom-sets*. Using the new names, the two conditions which qualify a graph as a category are the following.

Composition Given a pair of arrows $f : a \to b,\ g : b \to c \in \mathrm{Arr}(C),\ a, b, c \in \mathrm{Obj}(C)$, there exists an arrow $fg : a \to c \in \mathrm{Arr}(C)$. Moreover, composition is associative.

Identity For every object $b \in \mathrm{Obj}(C)$, there exists an *identity arrow* $1_b : b \to b \in \mathrm{Arr}(C)$ satisfying the compositions $f1_b = f$ and $1_b g = g$ (with f and g as above).

The identity arrow 1_c for a given object $c \in \mathrm{Obj}(C)$ is unique.

Because arrows are often functions of some sort, we use the terminology of functions to refer to their end points. If $f : a \to b \in \mathrm{Arr}(C)$ is an arrow starting at a and ending at b ($a, b \in \mathrm{Obj}(C)$), then the *domain* of f is a and the *codomain* of f is b. These are denoted $\mathrm{dom}(f) = a, \mathrm{cod}(f) = b$. We sometimes use the terminology of domains and codomains for graphs as well, when it is convenient.

Using the terminology of domains and codomains, we can define the hom-set[2] as

$$C(a, b) = \{f \mid f \in \mathrm{Arr}(C), \mathrm{dom}(f) = a, \mathrm{cod}(f) = b\}$$

Two arrows in the same hom-set are said to be *coterminal*. Likewise, two edges in the same edge-set are said to be coterminal.

There are also some basic definitions in graph theory which are worth explaining at the outset. A graph H is a *subgraph* of a graph G, denoted $H \subseteq G$, if $\mathrm{V}(H) \subseteq \mathrm{V}(G)$, and for all $v, w \in \mathrm{V}(H)$ we have $H(v, w) \subseteq G(v, w)$. A category D is a *subcategory* of C, denoted $D \subseteq C$, if $\mathrm{Obj}(D) \subseteq \mathrm{Obj}(C)$,

$$D(a, b) \subseteq C(a, b),\ \forall a, b \in \mathrm{Obj}(D),$$

and for each object $a \in \mathrm{Obj}(D)$ its identity $1_a \in \mathrm{Arr}(D)$ is the same as its identity in C.

A subcategory $D \subseteq C$ is *full* if, for all the objects $a, b \in \mathrm{Obj}(D)$, we have equal hom-sets $D(a, b) = C(a, b)$. Which is to say that, for all the objects a, b in $\mathrm{Obj}(D)$, all of the edges or arrows connecting a to b in C also connect a to b in D. We also say a subgraph is full if the analogous conditions are met.

A *path* in a graph G is an ordered set of edges $p_1 p_2 ... p_n$ such that $\mathrm{dom}(p_i) = \mathrm{cod}(p_{i-1})$. We say that the edges are *consecutive*, *i.e.* the end of the first arrow is the start of the next arrow, and so on. The *length* of a path is the number of edges in the path, and we can denote

[6].

[2]Other authors, such as Mac Lane, prefer the notation $\hom(a, b)$ or $\hom_C(a, b)$. The reader is warned in general that notation in category theory has not yet been well standardized. The conventions I use are an attempt to harmonize the usage of Mac Lane, Bergman, and Tilson.

a path p from vertex v to vertex w by $p : v \to w$. For any vertex v we define an *empty path* $1_v : v \to v$ with length 0. We can also compose paths $p : v \to w,\ q : w \to x,\ pq : v \to x$ (concatenating their edge lists) if the end of the first is the beginning of the second. If a path begins and ends at the same vertex v, then the path is called a *loop*.

With the notion of paths we can discuss the way in which various parts of a graph are connected. We say that v and w are *strongly connected* if there exist two paths, one from v to w *and* another from w to v. The strongly connected relationship divides a graph into *strongly connected components*, which are the sets of vertices which are mutually strongly connected.[3] A graph which has a single strongly connected component is said to be *strongly connected*.

Finally, we define the *product* $G \times H$ of two graphs G and H as follows:

$$\mathrm{V}(G \times H) = \mathrm{V}(G) \times \mathrm{V}(H)$$

$$G \times H((g, h), (g', h')) = G(g, g') \times H(h, h')$$

We can iterate this product to form the product of a set of graphs $\{X_b \mid b \in \beta\}$.

$$\prod\{X_b \mid b \in \beta\}$$

The product of categories is defined in the same way, with coordinate-wise composition [9, p.92]. The product of empty graphs is the graph with one vertex and one edge, while the product of empty categories is the category with one object and one arrow (the monoid **1**, as we will see).

We can also define the *coproduct* of categories $C \vee D = E$ with the following rules:

$$\mathrm{Obj}(E) = \mathrm{Obj}(C) \cup \mathrm{Obj}(D) \text{ (disjoint union)}$$

$$E(a, b) = C(a, b) \text{ if } a, b \in \mathrm{Obj}(C)$$

$$E(a, b) = D(a, b) \text{ if } a, b \in \mathrm{Obj}(D)$$

$$E(a, b) = \emptyset \text{ otherwise.[9, p.93]}$$

Considering categories as graphs is perhaps the most intuitive way to approach the subject. A category is simply a graph with identity arrows, and where any pair of consecutive arrows implies that there is a composite arrow. We now consider some examples of graphs and categories.

Figure 1 depicts two graphs, G and H. In graph G there is a path from v to w with the edge sequence $abcde$. We can eliminate the loops c and bde in the path to get the loop-free path a. G has one component, so it is a connected graph. It is not strongly connected, since there is no path starting at y and ending at v. We can consider the edge-sets; the edge-set between $G(v, w)$ includes two edges $\{a, f\}$.

The two graphs share a special relationship. Graph G is a subgraph of H, because it shares a subset of its vertices (in fact all its vertices), and every edge-set in G is a subset of the corresponding edge-set in H.

[3] Tilson uses the term "bonded" instead of "strongly connected", but we will prefer the latter.

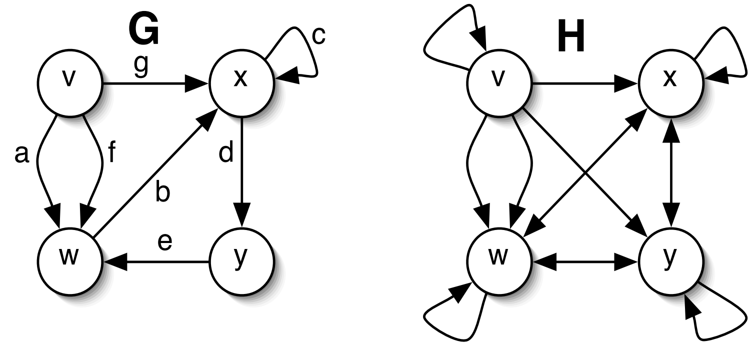

Figure 1: Two graphs.

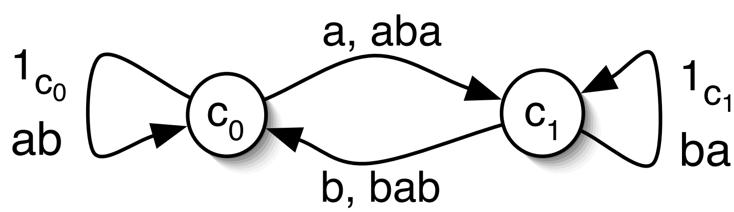

Figure 2: An example category.

Figure 2 depicts the category C. C consists of the objects c_0 and c_1, the identity arrows 1_{c_0} and 1_{c_1}, and the arrows $a, aba : c_0 \to c_1$, $b, bab : c_1 \to c_0$, $ab : c_0 \to c_0$, $ba : c_1 \to c_1$, with the multiplication by concatenation where $(ab)^2 = ab,\ (ba)^2 = ba$.

Although particular graphs are often intuitive to understand, and graph theory is well developed, it is often more difficult to consider graphs abstractly than other algebraic objects. There are other mathematical structures which are also related to categories, and which have advantages and applications different than graphs.

2.2 Axioms for Categories

From the intuitive notion of a category as a kind of graph, we now turn to the axiomatic definition of a category.[4] We leave aside for the time being some of the difficulties related to set theory (*i.e.* how to deal with sets of sets, especially in the infinite case), as these will be dealt with later.

Bergman's defines a category as a quadruple:

$$C = (\mathrm{Obj}(C), \mathrm{Arr}(C), \mu(C), \mathrm{id}(C))$$

where $\mathrm{Obj}(C)$ is a *collection* of elements, $\mathrm{Arr}(C)$ is a class of hom-sets, indexed by the pairs of elements in $\mathrm{Obj}(C)$,

$$\mathrm{Arr}(C) = (C(a, b))_{a,b \in \mathrm{Obj}(C)}$$

[4]Here I follow Bergman's §6.1, with slight changes in notation [1, p.147].

$\mu(C)$ is a set of binary operations

$$\mu(C) = (\mu_{abc})_{a,b,c \in \mathrm{Obj}(C)}$$
$$\mu_{abc} : C(a,b) \times C(b,c) \to C(a,c)$$

and $\mathrm{id}(C)$ is a set of elements

$$\mathrm{id}(C) = (1_a)_{a \in \mathrm{Obj}(C)}$$
$$1_a \in C(a,a)$$

such that the composition of maps μ_{abc} is associative, and

$$\forall f \in C(a,b), f1_a = f = 1_b f$$

where $a, b \in \mathrm{Obj}(C)$.

All of this amounts to the same thing as the definition of categories in terms of graphs. However the axioms are sometimes easier to apply in practise than graph theoretic notions. Just as before, we have the set of objects $\mathrm{Obj}(C)$ of the category, and the sets of arrows $\mathrm{Arr}(C)$ between pairs of objects. Recall that the hom-set $C(a,b)$, for any two objects, is the set of arrows starting in a and ending at b. Instead of speaking of the composition of arrows, we now speak of mappings in $\mu(C)$ which take the arrows in hom-set pairs to arrows in another hom-set. Finally, every object a has an identity arrow, which is a member of the hom-set $C(a,a)$.

We will usually take $\mathrm{Obj}(C)$ to be a set, or something like all algebraic objects of a given type, (*i.e.* all groups). But Bergman warns us this is a somewhat tricky matter. In the axioms $\mathrm{Obj}(C)$ is merely an "index set" [1, p.151]. A significant problem which arises from set theory is the question of self-referential sets, such as the set of all sets. Bergman augments the standard set theory, distinguishing between *small* and *large* sets, in such a way as to avoid this problem, and Mac Lane has a similar solution. We will make note of this in Section 2.5.

2.3 Monoids and Categories

A *monoid* $(M, +)$ is a mathematical structure which consists of a set of elements M, combined with a binary operation $+$ (*e.g.* addition, multiplication, concatentation, etc.), satisfying the following axioms:

Association The binary operation is associative: $\forall a, b, c \in M,\ a + (b + c) = (a + b) + c$

Identity M contains an *identity* element: $\exists 0_M \in M$ such that $\forall a \in M,\ 0_M + a = a = a + 0_M$

The monoid will be closed, by the nature of the binary operation. When the binary operation is clear, we refer to the monoid as M instead of $(M, +)$. A basic result from the theory of monoids is that the identity element is unique. If the binary operation $+$ is commutative

$$\forall a, b \in M, a + b = b + a$$

then we call M a *commutative monoid*.

Although the binary operation could be anything, there are two forms of notation for monoids: *additive* and *multiplicative*. So far we have used the additive notation, which is often more clear, but the multiplicative notation is usually more convenient. The axioms become:

Association $\forall a, b, c \in M,\ a(bc) = (ab)c$

Identity $\exists 1_M \in M$ such that $\forall a \in M,\ 1_M a = a = a 1_M$

We are often interested in the *units* of a monoid, which are the elements which have inverses: given $a \in M$, $a^{-1} \in M$ is the *inverse* of a if $aa^{-1} = 1_M = a^{-1}a$. The inverse of an element, if it has one, is unique. A *group* is different from a monoid only in requiring that all elements have an inverse. If we take all the units of any monoid M, and its binary operation, we will have a group.

Examples of monoids include the set of natural numbers under multiplication, and the integers under addition (which is also a group). Monoids are particularly useful in the study of formal languages. Given a set S of characters $\{a, b, c\}$, we can generate a *free monoid* S^* which is the set of all strings in S with the binary operation of concatenation, and the empty string 1 as the identity. For $S = \{a, b\}$ we have $S^* = \{1, a, b, ab, ba, ...\}$.

We say that a monoid N is a *submonoid* of a monoid M if the set of N is a subset of the set of M, and they share the same binary operation and the same identity. For example, the even integers under addition are a submonoid of the integers under addition, and they share the identity 0.

It is interesting to consider some other examples. Consider $M = (\{\emptyset, \{a\}, \{b\}, \{a, b\}\}, \cup)$, which is a monoid over sets using the union operation. Its identity is $\emptyset$, since the union of any set S with the empty set is S. We can describe the monoid with multiplication table 1.

Table 1: Multiplication for a Sample Monoid M.

$\cup$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$
$\emptyset$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$
$\{a\}$	$\{a\}$	$\{a\}$	$\{a,b\}$	$\{a,b\}$
$\{b\}$	$\{b\}$	$\{a,b\}$	$\{b\}$	$\{a,b\}$
$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$

Monoids are often used to represent strings. Take the monoid N with the set

$$\{0, 1, a, b, ab, ba, a^2, b^2\}$$

and with multiplication using the following rules:

(i) 0 multiplied by any element results in 0,

(ii) 1 is the identity,

(iii) all other multiplication is by concatenation, except that when the length of the concatenated string is greater than two, in which case the result is 0.

Table 2: Multiplication for a Sample Monoid N.

	0	1	a	b	ab	ba	a^2	b^2
0	0	0	0	0	0	0	0	0
1	0	1	a	b	ab	ba	a^2	b^2
a	0	a	a^2	ab	0	0	0	0
b	0	b	ba	b^2	0	0	0	0
ab	0	ab	0	0	0	0	0	0
ba	0	ba	0	0	0	0	0	0
a^2	0	a^2	0	0	0	0	0	0
b^2	0	b^2	0	0	0	0	0	0

Table 2 describes the multiplication of N.

When we consider what monoids are a bit more abstractly, we can quickly see the relationship to categories.[5] Take a mathematical object X (for concreteness one can think of a set). Now consider all of the mappings of X with itself. The set of all such homomorphisms will contain the identity, it will be closed under composition of homomorphisms, and the composition of homomorphisms is associative. (We have no guarantee that any homomorphism will have an inverse.) So the set of homomorphisms and the operation of composition give us a monoid. No matter what the object X is, we can always create a monoid in this way. Conversely, as Bergman states, "Every monoid M is isomorphic to a monoid of maps of a set into itself," which is the analog of Cayley's Theorem [1, p.148].

A category with one object X looks precisely like this monoid. For this category C we have

$$\mathrm{Obj}(C) = \{X\}$$

$$\mathrm{Arr}(C) = C(X, X)$$

i.e. the hom-set of homomorphisms of X with itself. We also know that the homomorphisms compose with each other, and that

$$1_X \in C(X, X) = \mathrm{Arr}(C)$$

A group is a one object category where every arrow in C has a two-sided inverse. Note that we are associating the monoid and the group with the *arrows*, not with the *object* of the category.

Since a category with one object is a monoid, there is a sense in which categories with multiple objects extend the theory of monoids [9, p.84]. This is Tilson's insight, which will be central to the later sections of this paper.

We define the *local monoids* of a category as its individual objects and their "local" arrows – that is the arrows which both begin and end at that object. For any object $a \in C$ we have $1_a \in C(a, a)$, and with this identity arrow the object counts as a monoid. The local monoid for a also includes all the arrows $C(a, a)$. When $C(a, a) = \{1_a\}$ then we say that the local monoid is *trivial*. We say that a category C is *locally trivial* when all of the local monoids are trivial in this way.

[5] Here I follow Bergman's §6.1 [1, p.147].

I mentioned earlier that these different approaches to category theory can be complementary. This is very clear when we consider the "local" nature of a category in terms of its local monoids, and the "global" nature of a category in terms of the graph which connects the local monoids. By looking at categories as both monoids and graphs, both local and global, we get a better sense of the big picture.

The use of the monoid approach to categories may not be as immediately obvious as the ones in terms of graphs or the axioms. Tilson, however, demonstrates how useful this connection is. We will see how he uses the theory of monoids to enrich category theory, and then uses category theory in turn to prove new results about monoids.

2.4 Examples of Categories

Some examples of categories are now in order. The following table is compiled from Mac Lane [6, particularly p.12]. The list could, of course, be much longer. The technical term "small" will be explained in the next section, but for now consider a small set to be a normal set.

Table 3: Examples of Categories.

Abbr.	Name	Objects	Arrows
0	Empty Category	None	None
1	One Object	a	1_a (identity arrows are assumed)
2	Two Objects	a, b	$f : a \to b$
3	Triangle	a, b, c	$f : a \to b, g : b \to c, fg : a \to c$
$\downarrow\downarrow$	Parallel	a, b	$f, g : a \to b$
Δ	Simplicial Category	all finite ordinals	all order preserving functions
Set	Sets	all small sets	all functions between them
$\mathbf{Set}_*$	Pointed Sets	small sets with a selected base point	all base-point preserving functions
$\mathbf{Ens}_V$	Ensembles over a set V	all sets within V	all functions between them
Cat	Category of Categories	all small categories	all functors
Mon	Monoids	all small monoids	all morphisms of monoids
Grp	Groups	all small groups	all morphisms of groups
Ab	Abelian Groups	all small (additive) Abelian groups	all morphisms between them
Rng	Rings	all small rings	all morphisms of rings
CRng	Commutative Rings	all small commutative rings	all morphisms of rings

The four simplest categories are: **0**, the empty category; **1** the category on one object a and its identity arrow 1_a; **2**, the category with two objects a, b, two identity arrows $1_a, 1_b$, and one arrow connecting them f; and **3**, the category with three objects a, b, c, their identity arrows $1_a, 1_b, 1_c$, and a commutative triangle of arrows $f, g, h = fg$. Another simple category, denoted by $\downarrow\downarrow$ and called "parallel," is similar to **2** except that there is a pair of arrows f, g from a to b. Figure 3 below depicts **1**, **2**, **3**, and $\downarrow\downarrow$.

It is worthwhile to provide an example of the product of categories. As we saw in section 2.1, the product of categories C and D is given by

$$\mathrm{Obj}(C \times D) = \mathrm{Obj}(C) \times \mathrm{Obj}(D)$$

$$C \times D((c, d), (c', d')) = C(c, c') \times D(d, d')$$

Figure 4 shows the product $\mathbf{2} \times \mathbf{2}$.

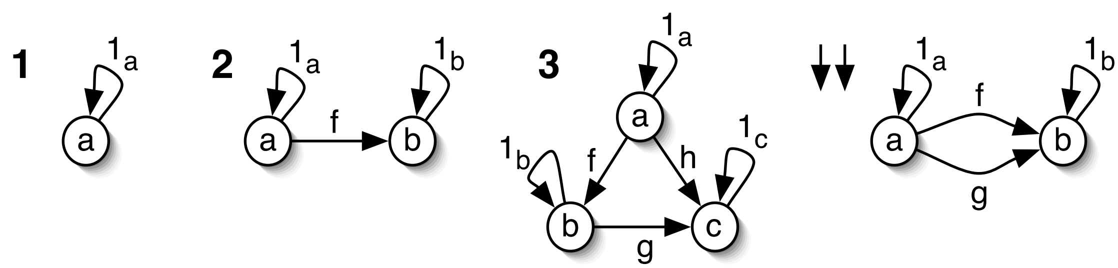

Figure 3: The four basic categories **1**, **2**, **3**, and ↓↓.

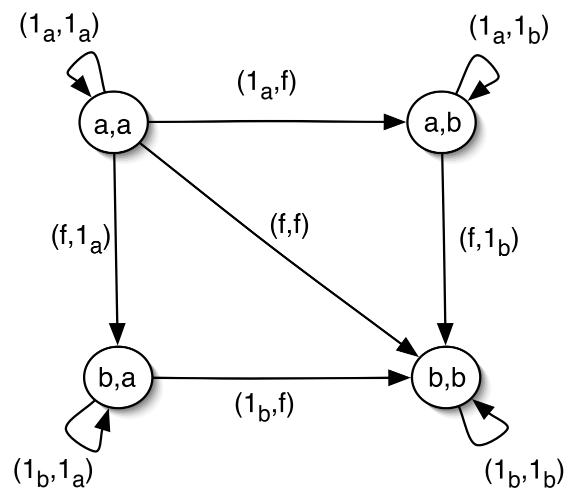

Figure 4: The category product $\mathbf{2} \times \mathbf{2}$.

Most of the other interesting categories take as their objects all of the algebraic objects of a given type, *i.e.* sets, monoids, or groups. The arrows are then some class of morphisms between the objects. But the notion of a category is a very general one. There are categories not just of algebraic objects, but of vector spaces, topological spaces, metric spaces and many other kinds of spaces, magmas, automata, and so on. The exceptional generality of category theory is one of its primary attractions for mathematicians.

2.5 Categories and Sets

Above I mentioned a problem with our characterization of categories as it stands. Bergman's axioms refer to a "collection" of elements, which has not been defined, and both he and Mac Lane see a problem with defining the object set of a category without further restriction. Our task in this section is to present the solution which Bergman and Mac Lane offer.

The problem with simply using sets in the definition of a category is the following. We want to use categories to explore algebraic objects, many of which (like monoids) are built upon sets. So we want to consider the set of *all* sets which meet certain conditions. As Mac Lane says, "This amounts to applying a comprehension principle: Given a property $\varphi(x)$ of sets X, form the set $\{x \mid \varphi(x)\}$ of all sets with this property" [6, p.21]. But this leads to a family of famous paradoxes due to Russell. For example, what if we define $\varphi(x)$ as the property "x is not a member of itself", and then consider the set $S = \{x \mid \varphi(x)\}$? Is S a member of S? If S is a member of S, then it must not have the property φ, and so S cannot be a member of S. If S is not a member of S, then it must have the property φ, and so S must be a member of S. Either way there is a contradiction.

The solution which both Bergman and Mac Lane employ is to limit the sets under consideration. They begin with the standard ZFC axioms for set theory, and add the concept of a *universe* in order to distinguish two kinds of sets. As Bergman defines it, a universe U is a set satisfying:

1. $X \in Y \in U \Rightarrow X \in U$
2. $X, Y \in U \Rightarrow \{X, Y\} \in U$
3. $X, Y \in U \Rightarrow X \times Y \in U$
4. $X \in U \Rightarrow \mathbf{P}(X) \in U$
5. $X \in U \Rightarrow (\bigcup_{A \in X} A) \in U$
6. $\emptyset \in U$
7. If $X \in U$ and $f : X \to U$ is a function, then $\{f(x) \mid x \in X\} \in U$.[1, p.162]

where $\mathbf{P}(X)$ is the power set of X, and $\emptyset$ is the null set. With reference to some U, we say that a set is *small* if it belongs to U, and we call an arbitrary set a *large* set. We can also call a category small if its set of objects is small, and large if its set of objects is large. Although Mac Lane does not, Bergman adds the following axiom to ZFC:

Axiom of Universes: Every set is a member of a universe.

With these tools in hand we can define the category of small sets, or the category of small groups. These are not themselves small, but they are well defined and avoid the problems mentioned above. Bergman's use of the word "collection" implies that the objects in the collection are small. Hereafter, if a universe is necessary then it will be assumed, without explicit mention.

Two more definitions connecting categories to sets may be helpful. A *discrete category* is one in which all of the arrows are identity arrows. So a discrete category is determined solely by its set of objects. When the objects of a category C are sets, and the arrows are functions, (or there is a faithful functor from C to **Set**) the category is called a *concrete category*.

2.6 Functors

Once we have multiple categories we can begin to ask questions about the relationships between them. A *functor* is essentially a morphism on categories. We define it in the following way.[6]

[6] In this section I follow Mac Lane [6, §1.3].

Let C and D be two categories. The functor $T : C \to D$ consists of two functions, one mapping the objects of C to the objects of D, and the other mapping the arrows of C to the arrows of D.[7] We call these the *object function* and the *arrow function*, and we use T to denote both of them (relying on the context to remove confusion). The arrow function must satisfy these two constraints:

$$T(1_c) = 1_{T(c)}$$

$$T(g \circ f) = T(g) \circ T(f)$$

where 1_x indicates the identity arrow for an object x, and $\circ$ indicates the composition of arrows.

Instead of speaking of the arrow function, it often makes more sense to talk about a family of functions, one for each hom-set in the domain. This is because we are usually more interested in the action on the hom-sets than on the global set of arrows. A functor is then an object function and a family of *hom-set functions*.

At this point it is convenient to introduce another form of notation, which is often preferable. In keeping with the notation used above for the composition of arrows in a graph, where $f : a \to b,\ g : b \to c$ becomes $fg : a \to c$, we can use the same convention for functions and functors. So if f, g are homomorphisms of monoids, $f : L \to M,\ g : M \to N$, we have $fg : L \to N$. If $a \in L$, and $f(a) = b \in M$, we can adopt the alternative notation $af = b$. Likewise for functors, $S : C \to D,\ T : D \to E$ produce $ST : C \to E$, and we can write the object functions as $cS = d$, and the arrow function as $fS = g$.

Adopting this new notation, the above requirements for the arrow function of a functor become

$$1_c T = 1_{cT}$$

$$(fg)T = fTgT$$

We will use this notation from now on for functors and morphisms, while using the $F(x)$ notation only for well established examples.

Just like functions, two functors can be composed. We do this by composing the two object functions and the two arrow functions (using the normal composition of functions). If *both* the object function and the arrow function of a functor are identity functions, then we call that functor an *identity functor*. We call a functor an *isomorphism* if *both* the object function and the arrow function are bijections. Equivalently, a functor is an isomorphism if is has an inverse functor – given functors $S : C \to D$ and $T : D \to C$, if ST and TS are identity functors, then $S = T^{-1}$ is the *two-sided inverse* of T.

Mac Lane gives the example of the power set functor $\mathbf{P} : \mathbf{Set} \to \mathbf{Set}$. For each set X, $\mathbf{P}(X)$ is the power set. For each arrow $f, g : X \to Y$ we have $f\mathbf{P} : \mathbf{P}(X) \to \mathbf{P}(Y)$. It is clear that $\mathbf{P}(1_X) = 1_{\mathbf{P}(X)}$ and $(fg)\mathbf{P} = f\mathbf{P}g\mathbf{P}$. So $\mathbf{P}$ is a functor.

We consider two more examples of functors. The first example is simple – $* : \mathbf{CRng} \to \mathbf{Grp}$, where $*$ takes a commutative ring K to the corresponding group of units K^*. The objects of **CRng** are commutative rings, and the arrows are ring morphisms, while the objects of **Grp** are groups, and the arrows are group morphisms. The object function of

[7] We can distinguish between covariant and contravariant functors, depending in the direction in which the images of the arrows act. Since we will only be concerned with covariant functors, the term "functor" will always refer to a covariant functor.

$*$ takes a ring to its group of units with the "multiplication" operation, while the arrow function takes ring homomorphisms to the group homomorphism on the "multiplication".

The second example is $\mathrm{GL}_n : \mathbf{CRng} \to \mathbf{Grp}$; the general linear functor (using $n \times n$ matrices) from the category of commutative rings to the category of groups. We begin by observing that, given any commutative ring K, we can construct the set of all non-singular $n \times n$ matrices with entries from K. This is a group under matrix multiplication, and we call it the general linear group $\mathrm{GL}_n(K)$. Given a ring homomorphism $f : K \to J$, we also have a group homomorphism $f\mathrm{GL}_n : \mathrm{GL}_n(K) \to \mathrm{GL}_n(J)$. Now we see that the object function of GL_n maps rings to groups, and the arrow function maps ring homomorphisms to group homomorphisms. So for any n, we have a functor $\mathrm{GL}_n : \mathbf{CRng} \to \mathbf{Grp}$.

We can define several different kinds of functor. A *forgetful* functor ignores some of the structure of its domain. For example, we can define a functor from **Grp** to **Set**, but because sets have less structure than groups the functor "forgets" some of the group structure of the groups it transforms, leaving a bare set.

A functor $T : C \to D$ is a *full* functor if for every arrow in D which connects the images of two objects from C, there is an arrow in C connecting those same objects. That is, T is full when

$$\text{if } \forall a, b \in \mathrm{Obj}(C), \text{ and } g \in D(aT, bT),$$
$$\text{then there is some } f \in \mathrm{Arr}(C), f : a \to b \text{ such that } fT = g.$$

More simply put, on each of the hom-sets in C a full functor is onto. Fullness is preserved by the composition of functors.

A *quotient* functor $Q : C \to D$ is similar to a full functor, except that it also requires that the object function between C and D be a bijection. A full functor may have an arbitrary object function, and so it can map any number of hom-sets onto a single hom-set. A quotient functor does more to preserve the structure of its domain.

We call a functor *faithful* if it preserves parallel arrows. So $T : C \to D$ is faithful when

$$\text{if } \forall a, b \in \mathrm{Obj}(C) \text{ and } f, g \in C(a, b),$$
$$\text{then } fT = gT : aT \to bT \Rightarrow f = g.$$

In other words, for each of the hom-sets in C a faithful functor is one-to-one. Composition of functors also preserves faithfulness.

We discussed the notion of a subcategory above, with reference to subgraphs. A category C is a subcategory of D if C contains some of the objects and the arrows of D, and the domains, codomains, identities, and composites of the arrows are preserved. We can speak of the *inclusion* or *embedding* functor, which maps C to D. The inclusion functor must always be a faithful functor. When the inclusion functor is full, we say C is a *full subcategory* of D. In this case the arrow function is an isomorphism, so we can specify the full subcategory in reference to the supercategory using its objects alone.

Table 4 below comes from Tilson, and summarizes the different kinds of functors [9, p.92].

2.7 Natural Transformations

The concept of a natural transformation is what Eilenberg and Mac Lane were trying to formalize in their 1945 paper [3]. A natural transformation is essentially a morphism of functors.

Table 4: Types of Functor, adapted from Tilson [9, p.92].

Type	Object Function	Hom-Set Functions
Isomorphism	Bijection	Bijection
Embedding	Injection	Injection
Faithful	Arbitrary	Injection
Quotient	Bijection	Surjection
Full	Arbitrary	Surjection

We define a *natural transformation* τ as follows.[8] Given two functors $S, T : C \to D$, $\tau : S \to T$ is a function assigning every object $a \in \mathrm{Obj}(C)$ to an arrow $a\tau \in \mathrm{Arr}(D)$, $a\tau : aS \to aT$, such that for any other arrow $f : a \to b$, $f \in \mathrm{Arr}(C)$, $a, b \in \mathrm{Obj}(C)$ the following *commutative diagram* is satisfied:

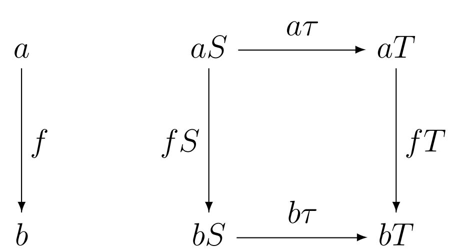

Figure 5: The commutative diagram for a general natural transformation.

We can consider each of the parts of the diagram separately. Each of the arrows in the commutative diagram corresponds to an arrow in the category D. The top arrow, $aS \xrightarrow{a\tau} aT$, connects the images of a under the functors S and T. The left arrow, $aS \xrightarrow{fS} bS$, is the image of the arrow $f \in \mathrm{Arr}(C)$, $f : a \to b$ under S. The right arrow, $aT \xrightarrow{fT} bT$, is the image of the same arrow $f \in \mathrm{Arr}(C)$ under the functor T. And the bottom arrow, $bS \xrightarrow{b\tau} bT$, connects the images of b under the functors S and T. In short, the diagram places a strong restriction on the natural transformation τ.

In more complicated cases, the commutative diagram will have to be applied more than once. See Figure 6. Of course, the diagrams may become arbitrarily complicated, but these two cases get across the important aspects of a natural transformation.

An example is in order. Consider the natural transformation $\det_K$, the determinant of an $n \times n$ matrix M with entries from the commutative ring K. The two functors under consideration are the ones we saw in the previous section: the functor $*$ from a ring to its group of units, and the functor GL_n from a ring to a group of $n \times n$ matrices, both of which operate from **CRng** $\to$ **Grp**. So starting with a ring K from the category **CRng**, we can construct the group of $n \times n$ non-singular matrices with entries in K, and the group of units K^*. The determinant of a non-singular matrix from $\mathrm{GL}_n(K)$ will always be a unit of the ring K, and so an element of K^*. Because the determinant is always calculated in the same

[8] I am making use of Mac Lane here [6, §1.4].

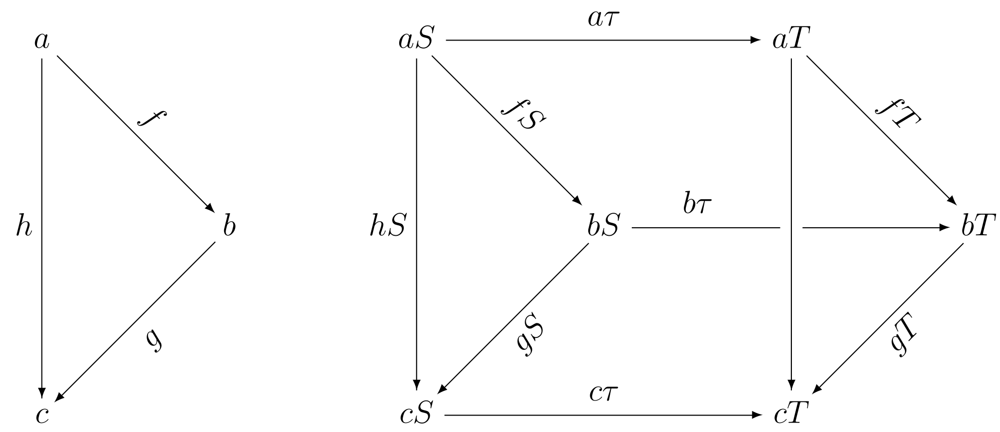

Figure 6: A more complicated commutative diagram.

way for any K, for any ring morphism $f : K \to J$ we will get a commutative diagram in Figure 7. Since $\det_K$ satisfies the commutative diagram, it is a natural transformation.

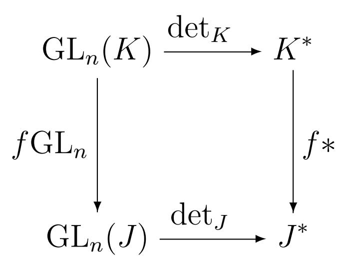

Figure 7: The commutative diagram for the natural transformation $\det_K$.

There are many other interesting examples of natural transformations, such as the natural transformation between a finite dimensional vector space and its double dual, which Eilenberg and Mac Lane use to introduce their 1945 paper [3, p.231-232].

This section has introduced categories from several different perspectives, and presented several of the basic results in the field. We are now ready to move on and explore more advanced topics from Tilson's work on monoids, categories, and varieties.

3 Division of Categories

We now turn to the paper "Categories as Algebra", by Bret Tilson [9]. Tilson makes use monoid theory to extend our understanding of categories, and then applies category theory to extend our understanding of monoids.

The first step is to adapt the definition of monoid division to allow for the division of categories. That leads the way to defining the derived category of a relational morphism, and to the Derived Category Theorem which is the chief result of this section. From there we follow Tilson's work on free categories and generators of categories, ending this section with a discussion of locally trivial categories. Applications of this work to varieties of monoids and categories will be the focus of the final section of this paper.

3.1 Division

Tilson's account is grounded on the extension of monoid division to a new division for categories, and this is where we will begin.[9]

Tilson makes use of what he calls a "relational morphism" is his definition, so it will be prudent to define this term before proceeding. We are familiar with the idea that a set *function* is a mapping $f : X \to Y$ from a set to another set, such that for all $x \in X,\ xf \in Y$. So for every element of X, f provides an element of Y. A set *relation* is a set-valued function, such that $R : X \to \mathbf{P}(Y)$. That is, a relation R maps an element $x \in X$ to a *set* of values in Y. Tilson calls a relation R *fully defined* when $\forall x \in X,\ xR \neq \emptyset$. That is, every x has some value under R. Further, he calls a relation R *injective* if $x, x' \in X,\ x \neq x'$ implies that $xR \cap x'R = \emptyset$. In other words, the images of distinct x's are disjoint.

There is an important connection between set relations and the product of sets. Let X and Y be sets, and Z be a subset of the cartesian product $Z \subseteq X \times Y$. Then associated with Z we have a set relation $R_Z : X \to Y$ such that for $x \in X$,

$$xR_Z = \{y \in Y \mid (x, y) \in Z\}$$

We also have the converse relationship; given a set relation $R : X \to Y$, we have a subset $R\# \subseteq X \times Y$, such that for $x \in X,\ y \in Y$,

$$R\# = \{(x, y) \mid y \in xR\}$$

So # is a bijection between set relations and subsets of the cartesian product. A relation is fully defined if and only if $R\#$ projects onto X in the first coordinate.

This leads Tilson to the definition of a "relational morphism" for monoids. A *relational morphism* $\varphi : M \lhd N$ is a fully defined set relation $\varphi : M \to N$, which satisfies these two conditions

$$\forall m, m' \in M,\ m\varphi m'\varphi \subseteq (mm')\varphi \tag{1}$$

$$1_N \in 1_M\varphi \tag{2}$$

One can check that $\varphi : M \lhd N$ implies that $\forall s \in M\ s\varphi \neq \emptyset$, and $\varphi\#$ is a submonoid of $M \times N$, and the converse. The relational morphism satisfies a "morphism-like" multiplication property (1), and identity property which says that 1_M has the identity of 1_N as a value. We can compose relational morphisms using the following rule:

$$y \in xfg \text{ if there exists some } z \text{ such that } z \in xf \text{ and } y \in zg.$$

Relational morphisms are closed under composition. In most ways, relational morphisms are the relational analogue of homomorphisms.

[9] In this section we follow Tilson's §A.2 [9].

Before extending this definition to categories, we stop for some examples. First of all, it should be clear that homomorphisms of monoids are relational morphisms. In this case, instead of the subset inclusion $m\varphi m'\varphi \subseteq (mm')\varphi$ we have the equality $m\varphi m'\varphi = (mm')\varphi$. Also, if we have an onto homomorphism of monoids $\theta : M \to N$, then $\theta^{-1} : N \lhd M$ will be a relational morphism. The identity condition is easy to show:

$$1_M\theta = 1_N \Rightarrow 1_M \in 1_N\theta^{-1}$$

We can also quickly derive the morphism property. Let $m' \in m\theta^{-1},\ n' \in n\theta^{-1}$. Then,

$$m'\theta = m,\ n'\theta = n$$

and since θ is a homomorphism we have

$$(m'n')\theta = m'\theta n'\theta = mn$$

Thus

$$m'n' \in (mn)\theta^{-1}$$

and since m and n are arbitrary, we have

$$m\theta^{-1}n\theta^{-1} \subseteq (mn)\theta^{-1}$$

as desired. We also know that θ^{-1} is fully defined, since θ is onto.

One more example of relational morphisms between monoids returns us to the examples from Section 2.3. Recall that N was the monoid over the set $\{0, 1, a, b, ab, ba, a^2, b^2\}$, while M was the monoid $(\{\emptyset, \{a\}, \{b\}, \{a, b\}\}, \cup)$. We define a relational morphism $\psi : N \to M$, which takes strings to the set of their letters. Thus we have:

$$1\psi = \emptyset$$

$$a^2\psi = a\psi = \{a\}$$
$$b^2\psi = b\psi = \{b\}$$
$$ab\psi = ba\psi = \{a, b\}$$

The value of 0ψ is somewhat trickier, however to have a relational morphism,

$$a^2a = 0 \Rightarrow a^2\psi \cup a\psi = \{a\} \subseteq 0\psi$$

$$b^2b = 0 \Rightarrow b^2\psi \cup b\psi = \{b\} \subseteq 0\psi$$
$$a^2b = 0 \Rightarrow a^2\psi \cup b\psi = \{a, b\} \subseteq 0\psi$$

Thus we conclude that $0\psi = \{\{a\}, \{b\}, \{a, b\}\}$ will give us a relational morphism. We could equally choose $0\psi = M$.

We can also define a relational morphism for categories. First, a relation of categories $R : C \to D$ includes an *object function* (not a relation) $R : \mathrm{Obj}(C) \to \mathrm{Obj}(D)$, and a family of *hom-set relations* where each relation r is such that

$$r : C(c, c') \to D(d, d')$$

and $d \in cR,\ d' \in c'R$.

As is the case with monoids, this is equivalent to $R\#$ being a subcategory of $C \times D$ with the projection to C a quotient functor. $R\#$ is a category such that

$$\mathrm{Obj}(R\#) = \{(c, d) \mid d \in cR\}$$

and hom-sets

$$R\#((c, d), (c', d')) = \{(f, g) \mid f \in C(c, c'),\ g \in D(d, d'),\ g \in fR\}$$

with component-wise multiplication, where $c, c' \in \mathrm{Obj}(C),\ d, d' \in \mathrm{Obj}(D)$.

A relational morphism of categories $\varphi : C \lhd D$ is a relation of categories $\varphi : C \to D$ such that each hom-set relation is fully defined, and

$$f\varphi g\varphi \subseteq (fg)\varphi \tag{3}$$

for all pairs $f, g \in \mathrm{Arr}(C)$ of consecutive arrows, and for all $a \in \mathrm{Obj}(C)$

$$1_{a\varphi} = 1_a\varphi \tag{4}$$

This preliminary work greatly simplifies the presentation of division for monoids and categories.

Tilson gives two definitions of monoid division. The first is the simplest. We say that a monoid M *divides* a monoid N, and we write $M \prec N$, if M is a homomorphic image of a submonoid of N.

The second definition involves relational morphisms. Given monoids M and N, we say that $M \prec N$ if and only if there exists an injective relational morphism $\varphi : M \lhd N$. In other words, φ is both fully defined and injective, so the image of every $m \in M$ is a non-empty set in N which is distinct from that of every other element of M. This is equivalent to the first definition since the inverse φ^{-1} will be a homomorphism taking a submonoid of N to M.

Division of monoids is a useful operation, because it defines a preorder. We can then say that monoids M and N are *equivalent* if $M \prec N$ and $N \prec M$, and we write $M \sim N$. Further, if $M \sim N$, and the monoids are finite, then M and N are isomorphic.

While the first definition of monoid division is more straightforward and doesn't involve the introduction of relations, the second definition is the one which applies best to the case of categories. Tilson defines a *division* as a relational morphism of categories where the hom-set relations are injective. It follows that any faithful functor is a division. Then he says that a category D *divides* a category C, written $D \prec C$, if there is a division $\varphi : D \prec C$. In other words, $D \prec C$ if there is an object function $\varphi : \mathrm{Obj}(D) \to \mathrm{Obj}(C)$, and a family of fully defined injective hom-set relations, one for each hom-set in D. As before, we say that two categories are equivalent ($D \sim C$) if $D \prec C$ and $C \prec D$. However, equivalent finite categories are not necessarily isomorphic. For example, $\mathbf{1} \sim \mathbf{2}$, but they are clearly not isomorphic.

As Tilson makes clear, the definitions of "division", "relational morphism", and "equivalence" for categories are the general cases, from which the special case of monoids easily follows (there is only one object, so the object function is always the identity) [9, p.96].

Tilson continues, showing that the first definition of monoid division has an analogue in the category case. For our purposes, however, the definition of category division given above is sufficient.

There are a few more results which we will find useful regarding category division. First, any category is divided by the empty category $\mathbf{0}$. The *trivial monoid* with one object divides any category except $\mathbf{0}$. We call a category *trivial* if every hom-set has at most one arrow.

Tilson states that,

Theorem 3.1 *For any category C, $C \prec \mathbf{1}$ iff C is trivial.*

Proof The proof is based on the following observation. If $R : X \to Y$ is a fully defined and injective set relation, then there must be an injective function r such that $r(x) \in R(x)$ for all $x \in X$. Hence the cardinality of X is less than or equal to the cardinality of Y, $\text{card}(X) \leq \text{card}(Y)$. Applied to the case of a division $\varphi : D \to C$, for $a, b \in \text{Obj}(D)$ we have the the same relation between the hom-sets:

$$\text{card}(D(a, b)) \leq \text{card}(C(\varphi(a), \varphi(b)))$$

So, if $C \prec \mathbf{1}$, then $\text{card}(C(a, b)) \leq 1$ for all $a, b \in \text{Obj}(C)$, and so C is trivial. Conversely, if $\text{card}(C(a, b)) \leq 1$ for all $a, b \in \text{Obj}(C)$, then there is a unique faithful functor $C \to \mathbf{1}$, and so $C \prec \mathbf{1}$. $\square$

It is interesting that the equivalence we defined above, where categories $C \sim D$, which we could call "divisional equivalence", is different than the notion of "natural equivalence" which involves two inverse functors connecting C and D. It follows from Theorem 3.1 that natural equivalence implies divisional equivalence, but not *vice versa*. It follows from this that $\mathbf{0}$ and $\mathbf{1}$ are the only trivial categories, up to divisional equivalence.

Also important is the fact that categories can divide monoids, and, as Tilson shows, every category divides some monoid [9, p.102-103]. We end this section with an example of a category which divides a monoid. We return to the category C, depicted in Figure 2, which consist of the objects c_0 and c_1, and the arrows $a, aba : c_c \to c_1$, $b, bab : c_1 \to c_0$, $1_{c_0}, ab : c_0 \to c_0$, $1_{c_1}, ba : c_1 \to c_1$, with the multiplication by concatenation where $(ab)^2 = ab$, $(ba)^2 = ba$. The monoid is $M = (\{\emptyset, \{a\}, \{b\}, \{a, b\}\}, \cup)$, as above. Since M is a monoid, we can think of it as a category with one object m, and with arrow set $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ where the identity arrow is $\emptyset$, and the composition is provided by the operation $\cup$. The division $\psi : C \to M$ takes each arrow to the set of its letters (identity arrows go to $\emptyset$):

$$\begin{gathered} c_0, c_1 \to m \\ 1_{c_0}, 1_{c_1} \to \emptyset \\ a \to \{a\} \\ b \to \{b\} \\ ab, aba, ba, bab \to \{a, b\} \end{gathered}$$

So the object function of ψ takes the objects $c_0, c_1 \in \text{Obj}(C)$ to M's unique object m, and the four hom-sets of C map to the single hom-set of M. Since each of the hom-sets of C contains two arrows, and these coterminal arrows always map to different arrows in M, all of the hom-set functions are injective.

3.2 The Derived Category of a Relational Morphism

The idea behind the derived category for a relational morphism $\varphi : M \to N$ is that, by focussing on the images of φ^{-1}, we can group the elements of M in such a way that the structure of φ becomes more clear.[10] This cannot be done with monoids alone, but it is possible using categories. The derived category was invented to generalize to monoids the kernel of a group homomorphism. For the group case, cosets of the kernel already do the desired grouping.

To prepare for the new concept, we return to relational morphisms of monoids. In the previous section we explained how, for a relational morphism $\varphi : M \lhd N$ between monoids M and N, we have a submonoid $\varphi\# = \{(m, n) \mid n \in m\varphi\} \subseteq M \times N$, where $m \in M$ and $n \in N$. Now we take $n \in N$, and $(m_0, n_0) \in \varphi\#$, and use them to form the function

$$[n, (m_0, n_0)] : n\varphi^{-1} \to nn_0\varphi^{-1}$$

$$m[n, (m_0, n_0)] = mm_0, \ m \in n\varphi^{-1}$$

If $n\varphi^{-1} = \emptyset$, then this is the empty function.

So $[n, (m_0, n_0)]$ is a function mapping between subsets of M, and its action in M corresponds to right-mutliplication by m_0. We know that

$$mm_0 \in (nn_0)\varphi^{-1} \text{ since } nn_0 \in m\varphi m_0\varphi \subseteq (mm_0)\varphi.$$

So the domain of $[n, (m_0, n_0)]$ is $n\varphi^{-1}$, its codomain is $(nn_0)\varphi^{-1}$, and it acts by right-multiplication by m_0.

If we also have $(m_1, n_1) \in \varphi\#$, and so $[nn_0, (m_1, n_1)]$ has domain $(nn_0)\varphi^{-1}$, then

$$[n, (m_0m_1, n_0n_1)] : n\varphi^{-1} \to nn_0n_1\varphi^{-1}$$

We can work out the composition

$$[n, (m_0, n_0)][nn_0, (m_1, n_1)] = [n, (m_0m_1, n_0n_1)].$$

The derived category of φ can now be defined. Its objects will be N, and its arrows will be these newly defined functions.[11] So we define the *derived category* D_φ of the relational morphism $\varphi : M \lhd N$ by

$$\text{Obj}(D_\varphi) = N$$

$$\text{Arr}(D_\varphi) = \{[n_1, (m, n)] : n_1 \to n_2 \ \mid \ (m, n) \in \varphi\#, \ n_1 n = n_2\}$$

Composition is as above. The identity for $n \in N$ is $[n, (1_M, 1_N)]$. The hom-set

$$D_\varphi(n_1, n_2) = \{[n_1, (m, n)] \mid (m, n) \in \varphi\#, \ n_1 n = n_2\}.$$

Tilson provides an alternative notation for the arrow $[n, (m_0, n_0)] \in D_\varphi(n, nn_0)$ as $[m_0, n_0] : n \to nn_0$. He also says that $[m_0, n_0] : n \to nn_0$ is *represented by* $(m_0, n_0) \in \varphi\#$.

From these definitions we derive the following lemma:

[10] Here we follow Tilson's §A.4 [9].

[11] Tilson defines the objects of D_φ to be the objects of the image $M\varphi$ in "Categories as Algebra" [9]. However, in "Categories as Algebra, II", Steinberg and Tilson instead use N, which improves the theory [7]. It turns out that both definitions give equivalent categories. This is the usage we adopt.

Lemma 3.2

$$[1_N, (m,n)] = [1_N, (m_0, n)] \text{ iff } m = m_0$$

Proof We see that $[1_N, (m,n)] = [1_N, (m_0, n)]$ implies that $m_1 m = m_1 m_0$ for all $m_1 \in 1_N\varphi^{-1}$. So when $m_1 = 1_M$, $m = m_0$. Conversely, if $m = m_0$ then $[1_N, (m,n)] = [1_N, (m,n)]$ trivially. □

Tilson shows that the derived category behaves as a kernel should [9, p.108]. We can state the relationship between relational morphisms and divisions in this way:

Result 3.3 *Let $\varphi : M \lhd N$ be a relational morphism. Then φ is a division if and only if $D_\varphi \prec \mathbf{1}$, i.e. D_φ is trivial.*

Proof If $\varphi : M \prec N$ is a division of monoids, then φ is an injective relation. Therefore, for all $n \in \mathrm{Obj}(D_\varphi) = N$ we have at most one element in $n\varphi^{-1}$. Thus there can be at most one map between two such sets, so every hom-set $D_\varphi(n, n_0)$ has at most one arrow. So D_φ is trivial, and by Theorem 3.1 $D_\varphi \prec \mathbf{1}$. Conversely, if D_φ is trivial and, $n \in m\varphi \cap m_0\varphi$ then $[1_N, (m,n)], [1_N, (m_0, n)]$ are in $D_\varphi(1_N, n)$ and so they must be equal by the triviality of D_φ. But then $m = m_0$, and so φ is a division. □

We can also show that the derived category is a proper generalization of the kernel of a group homomorphism. For simplicity, we will limit ourselves to the case of an onto group homomorphism.

Result 3.4 *If $\varphi : G \to H$ is an onto group homomorphism, then* $\mathrm{Ker}_\varphi \sim D_\varphi$.

Proof We begin with a claim that for all $g_0, g_1 \in G,\ h \in H$,

$$[h, (g_0, g_0\varphi)] = [h, (g_1, g_1\varphi)] \text{ iff } g_0 = g_1.$$

If $g_0 = g_1$ then the truth of the claim is clear. If $g \in h\varphi^{-1}$, then $gg_0 = gg_1$ and this implies that $g_0 = g_1$. So the claim holds.

Now we show that $D_\varphi(1_H, 1_H) \cong \mathrm{Ker}_\varphi$. We know that $[1_H, (g_0, g_0\varphi)] : 1_H \to 1_H$ iff $1_H g_0\varphi = 1_H$. So we can define $\psi : \mathrm{Ker}_\varphi \to D_\varphi(1_H, 1_H)$ by $g\psi = [1_H, (g, 1_H)]$, which is an onto homomorphism. By the claim above, it is also injective. Thus ψ is an isomorphism and $D_\varphi(1_H, 1_H) \cong \mathrm{Ker}_\varphi$. Therefore $\mathrm{Ker}_\varphi \prec D_\varphi$.

To show the converse, we choose for each $h \in H$ some $g_h \in h\varphi^{-1} \subseteq G$. Now we define a functor $\alpha : D_\varphi \to \mathrm{Ker}_\varphi$. Since Ker_φ is a group, it is also a monoid, and thus it is a category with a single object. So α has as its object function the unique mapping from the objects of D_φ to the single object of Ker_φ. The mappings on arrows are

$$([h, (g, g\varphi)] : h \to hg\varphi)\alpha = g_h g g_{hg\varphi}^{-1}$$

We can see that α is well defined, since by the claim we made there is only one representative of each arrow.

We show that α is a faithful functor (injective on hom-sets) by taking

$$x = [h, (g_1, g_1\varphi)],\ \ y = [h, (g_2, g_2\varphi)],$$

$$x, y : h \to h_1 \text{ where } h_1 = hg_1\varphi = hg_2\varphi$$

Then we find

$$x\alpha = y\alpha \Rightarrow g_h g_1 g_{h_1}^{-1} = h_g g_2 g_{h_1}^{-1} \Rightarrow g_1 = g_2 \Rightarrow x = y$$

So α is faithful.

Finally, we check that α is a functor. It meets the identity condition, since

$$[h_1, (1_G, 1_H)]\alpha = g_h 1_G g_{h1_H}^{-1} = g_h 1_G g_h^{-1} = 1_G$$

And the multiplication condition is met in the following way.

$$[h, (g_1, g_1\varphi)]\alpha[hg_1\varphi, (g_2, g_2\varphi)]\alpha = (g_h g_1 g_{hg_1\varphi}^{-1})(g_{hg_1\varphi} g_2 g_{hg_1\varphi g_2\varphi}^{-1}) =$$

$$= g_h(g_1g_2)g_{h(g_1g_2)\varphi}^{-1} = [h, (g_1g_2, (g_1g_2)\varphi)]\alpha \Rightarrow$$

$$[h, (g_1, g_1\varphi)]\alpha[hg_1\varphi, (g_2, g_2\varphi)]\alpha = [h, (g_1g_2, (g_1g_2)\varphi)]\alpha$$

Thus α is a faithful functor, so $D_\varphi \prec \mathrm{Ker}_\varphi$.

So $D_\varphi \sim \mathrm{Ker}_\varphi$ as required. □

3.3 The Derived Category Theorem

Our goal in this section is the Derived Category Theorem. We will proceed by establishing two lemmas, which will help us secure the proof of the main theorem. Our first lemma involves the relationships between divisions.

Lemma 3.5 *Given monoids M, N, L with relational morphisms as in Figure 8, such that*

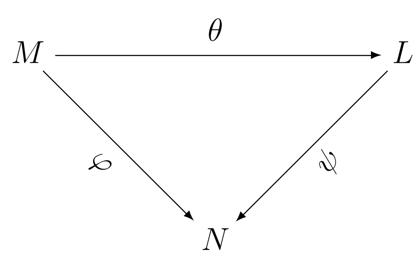

Figure 8: The relational morphisms of Lemma 3.5.

$\theta\psi = \varphi$ where θ is a division. Then $D_\varphi \prec D_\psi$.

Proof We define a new relational morphism $\bar{\theta} : D_\varphi \to D_\psi$. Its object function is the identity, and its hom-set functions are given by

$$[n, (m', n')]\bar{\theta} = \{[n, (l, n_0)] \mid \exists m'' \in M, n_0 \in N \text{ such that } [n, (m', n')] = [n, (m'', n_0)],\ l \in m''\theta\}$$

We want to show that $\bar{\theta}$ is a division.

For any $l' \in m'\theta$ we have $[n, (l', n')] \in [n, (m', n')]\bar{\theta}$. So $\bar{\theta}$ is fully defined.

Suppose that $[n, (l, n_0)] \in [n, (m', n')]\bar{\theta} \ \cap \ [n, (m'', n'')]\bar{\theta}$ with $nn' = nn_0 = nn''$. We are free to chose representatives such that $l \in m'\theta \cap m''\theta$. Since θ is a division, we have $m' = m''$. Thus the two arrows $[n, (m', n')]\bar{\theta}$ and $[n, (m'', n'')]\bar{\theta}$ are the same, and so $\bar{\theta}$ is injective.

Finally, we check that $\bar{\theta}$ is a relational morphism. The identity for any $n \in \mathrm{Obj}(D_\varphi)$ is $[n, (1_M, 1_N)]$. Since θ is a relational morphism, $1_L \in 1_M\theta$. So $[n, (1_L, 1_N)] \in [n, (1_M, 1_N)]\bar{\theta}$. Thus $\bar{\theta}$ meets the identity requirement. Let $[n, (l', n_0)] \in [n, (m', n')]\bar{\theta}$ and $[nn', (l'', n_1)] \in [nn', (m'', n'')]\bar{\theta}$. Without loss of generality, we can choose representatives such that $n_0 = n'$, $n_1 = n''$, $l' \in m'\theta$, and $l'' \in m''\theta$. Then $l'l'' \in m'\theta m''\theta \subseteq (m'm'')\theta$, and

$$[n, (l', n_0)][nn', (l'', n_1)] = [n, (l'l'', n'n'')] \in [n, (m'm'', n'n'')]\bar{\theta}$$

Thus $\bar{\theta}$ meets the multiplication requirement.

So we see that $\bar{\theta}$ is a division, and so $D_\varphi \prec D_\psi$ as required. □

The next step toward the Derived Category Theorem is to review the “wreath product”. Let V and N be monoids, and (following Tilson) use additive notation for the monoid V (and 0_V for the identity, although V is not necessarily commutative). We define V^N to be the monoid which has as its set all of the functions mappping N to V, and as its operation coordinate-wise addition. Thus the identity of V^N is the zero-function f_0.

We define a left action of N on V^N via

$$N \times V^N \to V^N, \ (n, f) \to {}^n f, \ n_0({}^n f) = (n_0 n)f$$

which statisfies

$${}^n(f + g) = {}^n f + {}^n g, \ {}^n({}^{n_0} f) = {}^{nn_0} f, \ {}^1 f = f, \ {}^n f_0 = f_0$$

The *wreath product* $V \circ N$ is the monoid with set $V^N \times N$ and binary operation

$$(f, n)(g, n_0) = (f + {}^n g, nn_0)$$

where $f, g \in V^N$ and $n, n_0 \in N$. The right-coordinate is a simple product in N, while the left-coordinate is influenced by N while operating in V^N. This influence is what distinguishes the wreath product from the normal cartesian product. The operation is clear when it is put in terms of matrices:

$$\begin{bmatrix} 1 & 0 \\ f & n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ g & n_0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ f + {}^n g & nn_0 \end{bmatrix}$$

The following are basic facts about the wreath product:

$$1 \circ N \approx N \text{ and } V \circ 1 \approx V$$

$$V \times N \prec V \circ N$$

$$\text{If } V \prec V' \text{ and } N \prec N', \text{ then } V \circ N \prec V' \circ N' \text{ .[9, p.113]}$$

Because the right-coordinate is well behaved, we can define a homomorphism $\pi : V \circ N \to N$ which is the projection function onto N, $\pi : V^N \times N \to N$. Tilson proves a second useful lemma which shows that the derived category of the projection is equivalent to V^N.

Lemma 3.6 *Let $\pi : V \circ N \to N$ be the projection morphism. Then $D_\pi \prec V^N$.*[12]

Proof First, observe that in an arrow $[h, ((f, n_0), n_0)]$ of D_π the second n_0 is redundant, so we will omit it by writing simply $[h, (f, n_0)]$. We define a relational morphism $\varphi : D_\pi \to V^N$. The object function is the unique function from the objects of D_π to the single object of the monoid V^N. The arrow relations are given by

$$[n, (f, n_0)]\varphi = \{{}^n g \mid [n, (g, n_1)] = [n, (f, n_0)]\}$$

We must check that φ is a division.

Let $[n, (f, n_0)] \in D_\pi$, then ${}^n f \in [n, (f, n_0)]\varphi$. Thus we see that φ is fully defined.

Next we show that φ is injective on the hom-sets of D_π. Suppose that

$$h \in [n, (f, n_0)]\varphi \ \cap \ [n, (g, n_1)]\varphi$$

with $nn_0 = nn_1$. We show that the arrows are the same. By changing the representatives of these arrows, we may assume that $h = {}^n f = {}^n g$. For any function l such that $(l, n) \in n\pi^{-1}$ we have

$$(l, n)(f, n_0) = (l + {}^n f, nn_0) = (l + h, nn_0) \tag{5}$$

but we know that $h = {}^n g$ and $nn_0 = nn_1$, so

$$(l + h, nn_0) = (l + {}^n g, nn_1) = (l, n)(g, n_1) \tag{6}$$

Comparing (5) and (6) we see that $[n, (f, n_0)]\varphi = [n, (g, n_1)]\varphi$, so φ is injective.

Finally, we check that φ is a relational morphism. It is clear that φ is a relation. For any object in $n \in D_\pi$, its identity is $[n, (f_0, 1_N)]$. We have $f_0 = {}^n f_0 \in [n, (f_0, 1_N)]\varphi$ which is the identity of V^N, so φ satisfies the identity requirement. As for the multiplication requirement, let $x = [n, (f, n_0)]$ and $y = [nn_0, (g, n_1)]$, and let $h \in x\varphi$, $l \in y\varphi$. Now we select the representatives such that $h = {}^n f$ and $l = {}^{nn_0} g$, so

$$h + l = {}^n f + {}^{nn_0} g = {}^n (f + {}^{n_0} g)$$

Since $xy = [n, (f + {}^{n_0} g, n_0 n_1)]$ we see that $h + l \in (xy)\varphi$ as required. So $x\varphi y\varphi \subseteq (xy)\varphi$.

Thus φ is a division, so $D_\pi \prec V^N$. $\square$

The last step before the main theorem is to observe the relation between the wreath product and division. For a monoid M, a *wreath product decomposition* is a division

$$\theta : M \prec V \circ N$$

By composing this division with the projection we find an associated relational morphism, $\varphi = \theta\pi : M \lhd N$. This was Tilson's motivation for introducing the relational morphism, and it will be used in the proof of the theorem.

Tilson's *Derived Category Theorem* establishes the connection between the derived category and wreath product decompositions of monoids[13]. It corresponds to the well known group theoretic fact that if $\varphi : G \to H$ is an onto homomorphism then $G \prec \mathrm{Ker}_\varphi \circ H$ [4].

[12] A slightly weaker version of Tilson's Proposition 5.1 [9, p.114].

[13] Tilson's Theorem 5.2 [9, p.116].

Theorem 3.7 (The Derived Category Theorem) (a) *Let $\varphi : M \lhd N$ be a relational morphism of monoids, and let V be a monoid satsifying $D_\varphi \prec V$. Then there is a division of monoids*

$$\theta : M \prec V \circ N$$

satisfying $\theta\pi = \varphi$ (where π is the projection morphism $\pi : V \circ N \to N$).

(b) *Let $\theta : M \prec V \circ N$ be a division of monoids, and let $\varphi = \theta\pi : M \lhd N$ be the associated relational morphism. Then*

$$D_\varphi \prec V^N.$$

Proof The proof of part (b) follows immediately from Lemmas 3.5 and 3.6 by applying the following diagram.

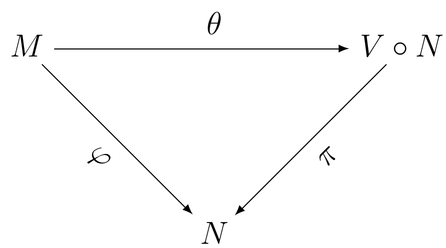

Figure 9: The commutative diagram for $\theta : M \prec V \circ N$.

To prove part (a), suppose that ψ is the division $\psi : D_\varphi \prec V$. For each $(m, n) \in \varphi\#$, we restrict V^N to a set of functions

$$F_{(m,n)} = \{f \in V^N \mid n_1 f \in [n_1, (m, n)]\psi,\ n_1 \in N\}$$

each of which is non-empty since ψ is fully defined. The identity f_0 of V^N belongs to $F_{(1_M, 1_N)}$ since ψ satisfies the identity condition.

Next we define a relation

$$\theta : M \to V \circ N,$$

$$m\theta = \{(f, n) \mid n \in m\varphi \text{ and } f \in F_{(m,n)}\}$$

Since both $m\varphi$ and $F_{(m,n)}$ are non-empty, $m\theta$ is non-empty. Also θ is defined such that $\varphi = \theta\pi : M \to N$. We now show that θ is (first) a relational morphism and (second) a division.

It is clear that θ is a relation of monoids. To show that θ is a relational morphism, we must show that the identity and multiplication conditions hold. Since $1_N \in 1_M\varphi$ and $f_0 \in F(1_M, 1_N)$, $(f_0, 1_N) \in 1_M\theta$. As for multiplication, we let $(f, n) \in m\theta$ and $(f', n') \in m'\theta$ and then take the product $(f, n)(f', m') = (f + {}^n f', nn')$. We know that $f \in F_{(m,n)}$ and $f' \in F_{(m',n')}$, so $\forall n_1 \in N$

$$n_1(f + {}^n f') = n_1 f + (n_1 n) f' \in [n_1, (m, n)]\psi + [n_1 n, (m', n')]\psi \subseteq [n_1, (mm', nn')]\psi$$

It follows that $f + {}^n f' \in F_{(mm',nn')}$ and $(f, n)(f', n') \in (mm')\theta$. So θ is a relational morphism.

To show that θ is a division, we need only show that it is injective. Suppose $(f, n) \in m\theta \cap m'\theta$. Then $n_1 f \in [n_1, (m, n)]\psi$ and $n_1 f \in [n_1, (m', n)]\psi$ for all $n_1 \in N$. But since ψ is a division, it is injective on any hom-set $D_\varphi(n_1, n_1 n)$, and so $\forall n_1 \in N,\ [n_1, (m, n)] = [n_1, (m', n)]$. In particular, when $n_1 = 1_N,\ [1, (m, n)] = [1, (m', n)]$, and as we saw in Lemma 3.2 this only occurs when $m = m'$. Therefore θ is injective, and so θ is the division which the theory says must exist. $\square$

We also have this corollary,

Result 3.8 *If $\varphi : G \to H$ is an onto group homomorphism, then $G \prec \mathrm{Ker}_\varphi \circ H$.*

Proof Any group homomorphism is a relational morphism. By Result 3.4 we know that $D_\varphi \prec \mathrm{Ker}_\varphi$. Thus by the Derived Category Theorem we have $G \prec \mathrm{Ker}_\varphi \circ H$ as required. $\square$

3.4 Congruences and Generators

In this section we explore a few more graph concepts which have applications to what we have been discussing, and will be useful in the future.[14]

First we will extend the notion of an equivalence relation to the case of graphs. An *equivalence relation* $\equiv$ on a graph G is a family of set equivalence relations for each of the edge-sets of the graph (recall that an edge-set is the graph analogue of a hom-set). The *equivalence classes* of $\equiv$ are the subsets of edges in each edge-set which are mutually equivalent. If we have vertices $v, w \in \mathrm{V}(G)$ and an edge $e : v \to w$, then we denote the equivalence class of e as $[e] : v \to w,\ [e] \subseteq G(v, w)$.

Given an equivalence relation, we can define a *quotient graph* $G/\equiv$, such that it shares the vertices of G, $\mathrm{V}(G/\equiv) = \mathrm{V}(G)$, and its edges are the equivalence classes of $\equiv$. We can also define the *quotient function* $\rho : G \to G/\equiv$ such that it maps the vertices of G to themselves, and the edges of G to their equivalence classes. Finally, we say a graph H is a *quotient* of a graph G if there is some equivalence relation $\equiv$ such that H is isomorphic to $G/\equiv$.

The next step is to extend this definition to categories. We call a graph equivalence relation $\equiv$ a *congruence* on a category C when for all b, b' coterminal and for all $a, c \in \mathrm{Arr}(C)$ such that abc is a valid composition,

$$b \equiv b' \Rightarrow abc \equiv ab'c$$

If $\equiv$ is a congruence, then $C/\equiv$ is a category with composition of equivalence classes:

$$[a][b] = [ab]$$

When this is the case, the same ρ as above is now called a *quotient morphism*. We adopt the expression $\equiv_1 \subseteq \equiv_2$ if we have $f \equiv_1 g \Rightarrow f \equiv_2 g$ for any pair of coterminal arrows in a category C.

A number of nice results follow from these definitions. We have

[14]In this section we follow Tilson's $ A.6 [9].

Result 3.9 *If $\equiv$ is a congruence on C, then $C/\equiv \prec C$.*[15]

Result 3.10 *Let C be a category, and let $\equiv_1$ and $\equiv_2$ be congruences on C with $\equiv_1 \subseteq \equiv_2$. Let $\rho_i : C \to C/\equiv_i$ be the quotient morphism induced by $\equiv_i$, $i = 1, 2$. Then there exists a quotient morphism $\theta : C/\equiv_1 \to C/\equiv_2$ satisfying $\rho_2 = \rho_1\theta$. Consequently, $C/\equiv_2 \prec C/\equiv_1$.*[16]

We can also derive a congruence $\equiv_\varphi$ on C from any category relational morphism $\varphi : C \to D$, using the rule

$$f \equiv_\varphi g \text{ iff } f \text{ and } g \text{ are coterminal and } f\varphi = g\varphi.$$

Instead of writing $C/\equiv_\varphi$, we usually write C/φ.

This leads to two more interesting results [9, pp.118-119]:

Result 3.11 *Let $\varphi : C \to D$ be a homomorphism. Then $C/\varphi \prec C$ and $C/\varphi \prec D$.*[17]

Result 3.12 *Let $\varphi : C \to D$ be a homomorphism, and let $\equiv$ be a congruence on C satisfying $\equiv \subseteq \equiv_\varphi$. Then φ factors uniquely through $C/\equiv$. That is, there exists a unique homomorphism $\theta : C/\varphi \to D$ satisfying $\varphi = \rho\theta$, where $\rho : C \to C/\equiv$ is the quotient morphism.*[18]

Finally, we come to the *free category* of a graph G, which we denote by G^*. We have,

$$\text{Obj}(G^*) = \text{V}(G)$$

$$\text{Arr}(G^*) = \{w \mid a \to b : a, b \in \text{Obj}(G),\ w \text{ is a path in } G\}$$

and the composition of arrows is the composition of paths by concatenation, discussed earlier. For every object a there is an identity arrow, which is the empty path $1_a : a \to a$.

In the case that G has a single object, G^* is a monoid and we say G^* is the *free monoid* over G.

Also interesting is the graph embedding of G in the category G^*. For any G^* we have such an embedding $i_G : G \to G^*$, which acts as the identity on objects, and takes each edge in G to the path of length 1 in G^* which corresponds to that edge.

Two results relate a free category to other categories, which Tilson calls "lifting properties" [9, p.120]:

Result 3.13 *Let G be a graph and let $\eta : D \to C$ be a quotient morphism of categories (that is, a full functor). If $\varphi : G^* \to C$ is a homomorphism, then there exists a homomorphism $\psi : G^* \to D$ such that $\varphi = \psi\eta$. In other words, Figure 10 commutes.*[19]

Result 3.14 *Let G be a graph and let $\varphi : G^* \to C$ be a homomorphism. If $C \prec D$ then there exists a homomorphism*

$$\psi : G^* \to D$$

that satisfies $\equiv_\psi \subseteq \equiv_\varphi$.[20]

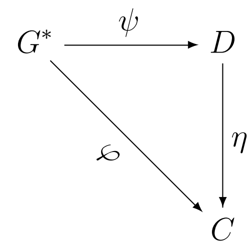

Figure 10: The commutative diagram for the lifting property.

While most examples of free categories will be infinite, it is possible to find finite ones. This will only occur is the graph G has no loops (of length greater than 0). Such a finite graph will have no path longer than the number of vertices. So we find,

Result 3.15 *The free category G^* is finite iff G is a finite loop-free graph.*[21]

From the free category of a graph, we come to the definition of a *generator* of a category. We say that a graph G *generates* a category C when C is a quotient of G^*, in particular $C \prec G^*$. If C is generated by a finite graph G, then it is *finitely generated*, and since $\mathrm{V}(G)$ is finite, $\mathrm{Obj}(C)$ is finite. It quickly follows that the empty graph generates the empty category, and the empty set (viewed as a one vertex graph with no edges) generates the smallest monoid **1**. Further,

Result 3.16 *If C is finitely generated, and $C \prec D$, where D is finite, then C is finite.*[22]

Given that C is a category generated by a graph G, we can find the smallest subcategory of C which contains G, called the *subcategory of C generated by G*. This leads to the following proposition.

Result 3.17 *Let G be a subgraph of a category of C. If no proper subcategory of C contains G, then X generates C.*[23]

3.5 Locally Trivial Categories

The last topic in this section is the class of locally trivial categories, which will be useful as we move forward.[24] Recall from 2.3 that the local monoids of a category are the monoids

[15]Tilson's Equation 6.3 [9, p.117].
[16]Tilson's Proposition 6.1 [9, p.118].
[17]Tilson's Proposition 6.2 [9, p.118].
[18]Tilson's Proposition 6.3 [9, p.118].
[19]Tilson's Proposition 6.4 [9, p.120].
[20]Tilson's Proposition 6.5 [9, p.120].
[21]Tilson's Equation 6.5 [9, p.121].
[22]Tilson's Equation 6.6 [9, p.121].
[23]Tilson's Proposition 6.6 [9, p.121].
[24]Here we follow §A.7 of Tilson's paper [9].

of local arrows $C(c, c)$, the local monoid is trivial when it contains only the identity, and a category is locally trivial when all its local monoids are trivial.

In this section we will be very interested in the category $\downarrow\downarrow$, depicted in Figure 3. The category $\downarrow\downarrow$ has two objects a, b with identity arrows, and two arrows $x, y : a \to b$. We can generalize this category, and call it $\downarrow\downarrow_X$, when we have a larger hom-set of arrows $\downarrow\downarrow_X (a, b) = X$.[25]

Tilson provides a series of results which show that all locally trivial categories divide various direct products of $\downarrow\downarrow$. The key theorem is this:

Theorem 3.18

Let C be a locally trivial category. Then

$$C \prec \prod\{\downarrow\downarrow_b |\ b \in \beta\}$$

where $\beta = \mathrm{Obj}(C) \times \mathrm{Arr}(C)$.[26]

Tilson builds up the proof of this theorem in a series of propositions. First he defines a *loop-free category* as a category generated by a loop-free graph. Since the generating graph is loop-free the only local loops in the free category are the identity arrows, and so a loop-free category is always locally trivial. The category $\downarrow\downarrow_X$ is generated by the graph which omits the identity arrows, and so it is loop-free.

The next step is to show that $\downarrow\downarrow_X$ divides a product of $\downarrow\downarrow_2 = \downarrow\downarrow$.

Result 3.19 *Let X be a set. Then*

$$\downarrow\downarrow_X \prec \prod\{\downarrow\downarrow_x |\ x \in X\}.^{27}$$

(We will omit the proofs here, since they are not crucial for our main point.)

Next, Tilson defines a preorder on vertices: For a graph G and vertices $v, w \in \mathrm{V}(G)$,

$$v \geq w$$

when there is a path $p : v \to w$ in G. We write $v > w$ when $v \geq w$ but not $v \geq w$ occurs. Using the terminology of categories,

$$\text{For } a, b \in \mathrm{Obj}(C),\ a \geq b \text{ iff } C(a, b) \neq \emptyset.$$

When C is generated by G, then these conditions are the same. Then we find that

Result 3.20 *Let C be a locally trivial category. Then C is loop-free if and only if the inequality $\geq$ is a partial order for C.*[28]

[25] Tilson calls these categories $\mathbf{A}_2$ and $\mathbf{A}_X$.

[26] Tilson's Theorem 7.1 [9, p.122].

[27] Tilson's Lemma 7.2 [9, p.123].

[28] Tilson's Lemma 7.3 [9, p.123].

Tilson uses this preorder to prove Theorem 3.18 in the special case of loop-free categories. But he also uses it to define the *strongly connected* (or *bonded*) components of a graph.[29] We saw in Section 2.1 that these components of a graph are the sets of vertices such that there is a path between every pair of vertices in the set. Tilson denotes this relation by $\sim$, and notes that the strongly connected components are the equivalence classes of $\sim$. A graph is strongly connected if it has a single strongly connected component. He presents this result about strongly connected components.

Result 3.21 *Let C be a locally trivial category. Then the strongly connected components of C are trivial categories, and C is equivalent to a loop-free subcategory of C.*[30]

The last concept that Tilson needs to prove the key theorem is the "transition edge". When $v, w \in \mathrm{V}(G)$, $e \in \mathrm{E}(G)$, $e : v \to w$, and $v > w$ then e is a *transition edge* from the strongly connected component of v to the strongly connected component of w. Since there is no path from w to v, a transition edge can only occur once in any path, and cannot occur in a loop.

Using this idea, we can define the *strongly connected normal form* (or the *bonded normal form* as Tilson calls it) [9, p.127]. For any path q, we can break q into paths (u_i) inside strongly connected components, and transition edges (t_j).

$$q = u_0 t_1 u_1 ... u_{n-1} t_n u_n, \ n \geq 0$$

Further, we can define a function τ which takes a path to its set of transition edges. Thus $\tau(q) = \{t_1, ..., t_n\}$. With this we define a new equivalence relation $\equiv_\tau$ such that

$$q \equiv_\tau q' \text{ iff } q\tau = q'\tau.$$

This leads to a pair of new results:

Result 3.22 (a) *The equivalence relation $\equiv_\tau$ is a congruence relation on G^*, and $G^*/\equiv_\tau$ is locally trivial. If G is finite, then so is $G^*/\equiv_\tau$.*
(b) *Let $\equiv$ be a congruence on G^* with $G^*/\equiv$ locally trivial. Then $\equiv_\tau \subseteq \equiv$.*[31]

The division of categories, and the derived category of a relational morphism, have allowed us to learn more about the structure of categories and monoids. It is now time to apply these result to discover more about varieties of monoids and categories.

4 Varieties of Categories

We move from the consideration of categories and monoids to a higher level of abstraction: "varieties" of categories and monoids. A variety is a collection which is closed under products and division, and Tilson's results from previous section allow him to describe in greater detail the shape that varieties of categories have.

[29] Tilson uses the word "bonded", but we will prefer the more common "strongly connected" in most cases.
[30] Tilson's Corollary 7.6 [9, p.126].
[31] Tilson's Proposition 7.7 [9, p.127].

In this section we will move more quickly, focusing on the results rather than the proofs. Our final goal will be the Strongly Connected Component Theorem, which tells us about the relationship between a category and the product of its components. Our first goal is to describe varieties of monoids and of categories, along with some basic results. Then we turn to what Tilson calls "laws," which are part of the graph structure of categories, and then "path equations," which can be used to form laws. Finally we come to the Strongly Connected Component Theorem.

The methods that Tilson uses to approach varieties include many of the results from the previous section. They also draw on the graph structure of categories, which gives us different insights into the workings of category theory.

4.1 Varieties of Monoids and Categories

We begin[32] by defining a *variety of monoids* as a collection $\mathbf{V}$ of monoids which satisfies the following:

Products The direct product of any set of monoids in $\mathbf{V}$ belongs to $\mathbf{V}$,

Divisors If $M \in \mathbf{V}$ and $N \prec M$, then $N \in \mathbf{V}$.

In short, a variety is a collection which is closed under direct products and division. Since we have defined both these operations for categories as well as monoids, we can define a *variety of categories* in exactly the same way.

In the case of categories, a variety is also closed under its coproducts [9, p.129]. In fact, a category is always isomorphic to the coproduct of its connected components, and

Result 4.1 *A variety of categories is generated by its connected members.*[33]

The one element monoid $\mathbf{1}$ is contained in all varieties of monoids, and the smallest variety of monoids is $\{\mathbf{1}\}$. We can speak of the *variety generated by* a set X of monoids or categories, denoted (X), which is the smallest variety which contains X. It is also true that the monoid $\mathbf{1}$ is contained in all varieties of categories, and that the smallest variety of categories is the one generated by $\mathbf{1}$. We call this the *trivial* variety, and denote it by $\mathbf{1_C}$.

We can use monoids to generate varieties categories more generally. If $\mathbf{V}$ is a variety of monoids, then it generates a variety of categories $\mathbf{V_C}$. Then if a category D divides a monoid $M \in \mathbf{V}$, we know that $D \in \mathbf{V_C}$.

Not all varieties of categories are generated by monoids, however. The collection of locally trivial categories $\mathcal{L}\mathbf{1}$ is such an example. The following result shows the connection between $\mathcal{L}\mathbf{1}$ and other varieties, particularly that it is the smallest variety other than $\mathbf{1_C}$.

Result 4.2 *Let* $\mathcal{L}\mathbf{1}$ *be the variety of all locally trivial categories.*
(a) $\mathcal{L}\mathbf{1} = (\downarrow\downarrow)$
(b) *Let* $\mathbf{W}$ *be a variety of categories. Then*

$$\mathcal{L}\mathbf{1} \subseteq \mathbf{W} \text{ iff } \mathbf{W} \neq \mathbf{1_C}$$

(c) *Let* $\mathbf{V}$ *and* $\mathbf{V}'$ *be non-trivial varieties of monoids satisfying* $\mathbf{V} \cap \mathbf{V}' = \{\mathbf{1}\}$*. Then*

[32] Here we follow Tilson's section B.8 [9].
[33] Tilson's Equation 8.6 [9, p.130].

$$\mathcal{L}\mathbf{1} = \mathbf{V_C} \cap \mathbf{V'_C}.$$ [34]

We now return to the wreath product, and define the wreath product of varieties. For varieties of monoids $\mathbf{V}$ and $\mathbf{W}$, the wreath product $\mathbf{V} \circ \mathbf{W}$ is the variety generated by

$$\{V \circ W \mid V \in \mathbf{V},\ W \in \mathbf{W}\}$$

Then we say

$$M \in \mathbf{V} \circ \mathbf{W} \text{ iff } M \prec V \circ W \text{ for some } V \in \mathbf{V} \text{ and } W \in \mathbf{W}.$$

The question is then how best to test when $M \in \mathbf{V} \circ \mathbf{W}$. Using the derived category, Tilson changes this into a question in terms of categories.

Theorem 4.3 *Let $\mathbf{V}$ and $\mathbf{W}$ be varieties of monoids. Then*

$$M \in \mathbf{V} \circ \mathbf{W}$$

iff there exists a relational morphism $\varphi : M \lhd N$ satisfying

$$D_\varphi \in \mathbf{V_C} \text{ and } N \in \mathbf{W}.$$ [35]

This allows us to make the following statements about the problem of whether $M \in \mathbf{V} \circ \mathbf{W}$:

Result 4.4 *Let M be a monoid and let $\mathbf{W}$ be a variety of monoids. Then the following statements are equivalent:*
(i) *There exists a relational morphism $\varphi : M \lhd N$ with $D_\varphi \in \mathcal{L}\mathbf{1}$ and $N \in \mathbf{W}$;*
(ii) *$M \in \mathbf{V} \circ \mathbf{W}$ for every non-trivial variety $\mathbf{V}$ of monoids;*
(iii) *$M \in \mathbf{V}_1 \circ \mathbf{W}$ and $M \in \mathbf{V}_2 \circ \mathbf{W}$ for some varieties $\mathbf{V}_1, \mathbf{V}_2$ with $\mathbf{V}_1 \cap \mathbf{V}_2 = \{\mathbf{1}\}$.*[36]

These results combine to transform the question above, in terms of monoids, into the following question about categories:

Problem Let C be a category and let $\mathbf{V}$ be a variety of monoids. Determine whther or not C belongs to $\mathbf{V_C}$.

This problem can easily be solved when a variety is "local", which Tilson defines in this way. First we consider the collection of all the categories with local monoids in a variety of monoids $\mathbf{V}$, denoted $\ell\mathbf{V}$,

$$\ell\mathbf{V} = \{C \in \mathbf{Cat} \mid C(c) \in \mathbf{V} \text{ for all } c \in \mathrm{Obj}(C)\ \}$$

We then say that $\mathbf{V}$ is it local if $\mathbf{V_C} = \ell\mathbf{V}$.

From this definition follow the result,

[34] Tilson's Theorem 8.1 [9, p.130].
[35] Tilson's Theorem 8.2 [9, p.131].
[36] Tilson's Proposition 8.3 [9, p.131].

Result 4.5 *Let C be a category and let $\mathbf{V}$ be a variety of monoids. If $\mathbf{V}$ is local, then*

$$C \in \mathbf{V_C} \text{ iff } \{C(c) \mid c \in \mathrm{Obj}(C)\} \subseteq \mathbf{V}.$$ [37]

Tilson shows that any variety constituted entirely of groups is local. A result of Simon shows that the variety of idempotent and commutative monoids is local, see Eilenberg [2]. And Thérien and Weiss show that the variety of commutative monoids is not local [8].

The next section provides tools which help us determine when a variety of monoids is local, and thus to answer our question about the wreath product of varieties.

4.2 Laws and varieties

Tilson applies work by Birkhoff on varieties of monoids to the case of category varieties.[38] Birkhoff established that one can completely specify a variety of monoids by the equations which the members satisfy. Tilson finds an analogue of these equations in category "laws".

He defines a *law* $(G; p = q)$ as a pair of coterminal paths p, q in a non-empty graph G. We say that a category C *satisfies a law* if every relational morphism $\varphi : G^* \to C$ satisfies $p\varphi = q\varphi$.

Figure 11 depicts the graphs for two examples of laws: $(L_1; pq = qp)$ and $(L_2; prq = qrp)$.

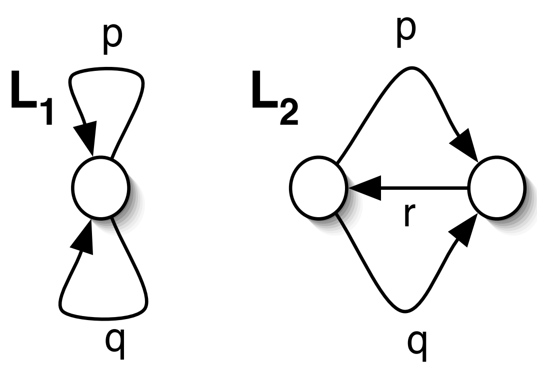

Figure 11: Two laws.

Two results show how laws relate to the product and division of categories:

Result 4.6 *If every category in a set $\{C_b \mid b \in \beta\}$ satisfying a law L, then the product $\prod\{C_b \mid b \in \beta\}$ satisfies L.*[39]

Result 4.7 *Let C satisfy a law L. If $D \prec C$, then D satisfies L.*[40]

Now we extend these definitions. Let $\mathbf{L}$ be a collection of laws. We say C satisfies $\mathbf{L}$ if it satisfies every law $L \in \mathbf{L}$. We define the variety $\mathbf{V}(\mathbf{L})$ as the set of categories which satisfy

[37] Tilson's Proposition 8.5 [9, p.134].
[38] We are following Tilson's §B.9 [9].
[39] Tilson's Proposition 9.1 [9, p.135].
[40] Tilson's Proposition 9.2 [9, p.135].

$\mathbf{L}$. Because the satisfaction of laws is maintained under products and division, it is clear that $\mathbf{V}(\mathbf{L})$ is a variety. We say that a variety $\mathbf{V}$ is *defined* by laws $\mathbf{L}$ if $\mathbf{V} = \mathbf{V}(\mathbf{L})$.

We can establish that all varieties are defined by a collection of laws, but first we must return to the topic of congruences. For a given graph G, consider all congruences on G^*, Cg_G. For a variety $\mathbf{V}$ we are interested in the congruences

$$\mathrm{Cg}_G(\mathbf{V}) = \{\equiv_\rho \in \mathrm{Cg}_G \mid G^*/\rho \in \mathbf{V}\}$$

That is, the congruences such that the quotient G^*/ρ belongs to the variety $\mathbf{V}$. With this Tilson defines a new congruence $\equiv_{\mathbf{V}}$ on G^*:

$$p \equiv_{\mathbf{V}} q \text{ iff } p \equiv_\rho q \text{ for all } \equiv_\rho \in \mathrm{Cg}_G(\mathbf{V}).$$

That is, $\equiv_{\mathbf{V}}$ is the smallest equivalence in the set $\mathrm{Cg}_G(\mathbf{V})$.

This leads to two relevant results:

Result 4.8 *Let G be a graph, let $\equiv_\rho$ be a congruence on G^*, and let $\mathbf{V}$ be a variety. Then*

$$\equiv_\rho \in \mathrm{Cg}_G(\mathbf{V}) \text{ iff } \equiv_{\mathbf{V}} \subseteq \equiv_\rho$$

In particular, $\equiv_{\mathbf{V}}$ is the unique minimal member of $\mathrm{Cg}_G(\mathbf{V})$.[41]

Result 4.9 *Let $\varphi : G^* \to C$ be a relational morphism with $C \in \mathbf{V}$. Then $\equiv_{\mathbf{V}} \subseteq \equiv_\varphi$.*[42]

Tilson defines the quotient $G^*/\equiv_{\mathbf{V}}$ as the *free category over G relative to* $\mathbf{V}$, and denotes it as $G^*/\mathbf{V}$. We can also use the congruence $\equiv_{\mathbf{V}}$ to define a set of laws from G and $\mathbf{V}$:

$$\mathbf{L}(G, \mathbf{V}) = \{(G; p = q) \mid p \equiv_{\mathbf{V}} q\}$$

Two more results come from this:

Result 4.10 *If $C \in \mathbf{V}$ and C is generated by G, then C is a quotient of $G^*/\mathbf{V}$. Consequently, $C \prec G^*/\mathbf{V}$.*[43]

Result 4.11 *Let $\mathbf{V}$ be a variety of categories and let C be a category. Then the following are equivalent:*
(i) *$C \in \mathbf{V}$;*
(ii) *C satisfies $\mathbf{L}(G, \mathbf{V})$ for every graph G;*
(iii) *C satisfies $\mathbf{L}(G, \mathbf{V})$ for some graph G that generates C.*[44]

We say that $\mathbf{L}_{\mathbf{V}} = \bigcup\{\mathbf{L}(G, \mathbf{V})\}$. From this last result, it follows that

Result 4.12 *The variety $\mathbf{V}$ is defined by the laws $\mathbf{L}_{\mathbf{V}}$. Consequently, every variety of categories is defined by laws.*[45]

[41] Tilson's Proposition 9.4 [9, p.136].
[42] Tilson's Corollary 9.5 [9, p.136].
[43] Tilson's Corollary 9.6 [9, p.137].
[44] Tilson's Proposition 9.7 [9, p.137].
[45] Tilson's Proposition 9.8 [9, p.137].

Returning to our two examples of laws, consider the variety of all commutative monoids **Com**. The variety of local monoids of **Com** is denoted ℓ**Com**. The variety of categories generated by **Com** is $\mathbf{Com_C}$. ℓ**Com** is defined by the law $(L_1; pq = qp)$, while $\mathbf{Com_C}$ is defined by $(L_2; prq = qrp)$ [8].

Tilson's laws are designed to be the equivalent of Birkhoff's equations, and they make use of the systems of paths which exist inside a category. As we saw above, more can be done with paths, connectivity, and components for a graph. The next section makes use of these graph structures.

4.3 Path Equations

The previous work on laws gives rise to a more specific case called a "path equation". Tilson makes use of this concept to explore the behaviour of strongly connected components of categories, and to further support the connection between laws and varieties.[46]

Consider the minimal graph necessary to *support* a law $(G; p = q)$. That is, the graph G such that every edge of $\mathrm{E}(G)$ appears in p or q. In this case we write $p = q$, and call the law a *path equation.* When the graph G is strongly connected, we call it a *strongly connected path equation.*

The key result of this section is the following:

Theorem 4.13 *Every variety of categories is defined by path equations. If the variety is non-trivial, it is defined by strongly connected path equations.*[47]

The proof of this theorem takes Tilson through a number of other theorems and propositions. First, he calls a law L *trivial* if $\mathbf{V}(L) = \mathbf{1_C}$. Second, he calls two collections of laws $\mathbf{L}$ and $\mathbf{L}'$ *equivalent* if $\mathbf{V}(\mathbf{L}) = \mathbf{V}(\mathbf{L}')$. Then he proves that

Theorem 4.14 *Every non-trivial law is equivalent to a finite number of strongly connected path equations.*[48]

Which is to say that we can build laws (pairs of paths) out of a finite number of strongly connected path equations. This leads to the result:

Result 4.15 *If C satisfies a law L, then C satisfies every derivative of L.*[49]

The proof of Theorem 4.14 then proceeds as follows: Let H be the support of the law $L = (G; p = q)$. If H has a transition edge other than those in p and q, then $\mathbf{V}(\mathrm{L})$ can be shown to equal $\mathcal{L}\mathbf{1}$. Otherwise, we break the paths p and q into their strongly connected normal forms, and create a collection of laws $\mathbf{L}$ for the $p_k = q_k,\ k = 0, ..., i$. Tilson shows that a category C satisfies L if and only if C satisfies $\mathbf{L}$, since every law in $\mathbf{L}$ is a derivative of L.

We indicate how Theorem 4.13 follows from Theorem 4.14. In the simplest case, the trivial variety $\mathbf{1_C}$ is defined by the path equation $a = b$. In the non-trivial case, we know that every non-trivial law is defined by a finite number of strongly connected path equations, and every variety is defined by its laws. Thus we have Theorem 4.13.

[46] Here we explore §B.10 of "Categories as Algebra" [9].
[47] Tilson's Theorem 10.1 [9, p.139].
[48] Tilson's Theorem 10.2 [9, p.139].
[49] Tilson's Equation 10.5 [9, p.141].

4.4 The Strongly Connected Component Theorem

Our goal is still to solve the problem presented earlier:

Problem Let C be a category and let $\mathbf{V}$ be a variety of monoids. Determine whther or not C belongs to $\mathbf{V_C}$.

We now know that we can answer this question in terms of the strongly connected components of C. The solution to our problem is the *Strongly Connected Component Theorem*. We will only sketch the proof here.

Theorem 4.16 (The Strongly Connected Component Theorem) *Let* $\mathbf{V}$ *be a non-trivial variety of categories. Then* $C \in \mathbf{V}$ *iff the strongly connected components of* C *belong to* $\mathbf{V}$.[50]

Proof Since $\mathbf{V}$ is not trivial, we know by Theorem 4.13 that it is defined by strongly connected path equations. Let $p = q$ be a strongly connected path equation on a graph G, and $\varphi : G^* \to C$ a functor. Since the image of a strongly connected category is strongly connected, $p\varphi, q\varphi$ are contained in a strongly connected component of C. Thus V satisfies $(G; p = q)$ if and only if each strongly connected component of C does. □

From basic definitions and results of category theory, through Tilson's work on division and the derived category, we have arrived at some interesting results about varieties of categories and the relationships between categories and their components. This has only been a brief introduction; category theory in general, and Tilson's work in particular, have much more to teach us about algebra.

[50]Tilson's Theorem 11.4 [9, p.145].

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